

$R_{1-tt}^{\mathcal{N}}(\text{NP})$ Distinguishes Robust Many-One and Turing Completeness

Edith Hemaspaandra*
Department of Mathematics
Le Moyne College
Syracuse, NY 13214, USA

Lane A. Hemaspaandra†
Department of Computer Science
University of Rochester
Rochester, NY 14627, USA

Harald Hempel‡
Institut für Informatik
Friedrich-Schiller-Universität Jena
07740 Jena, Germany

September, 1996; Revised July 15, 1997

Abstract

Do complexity classes have many-one complete sets if and only if they have Turing-complete sets? We prove that there is a relativized world in which a relatively natural complexity class—namely a downward closure of NP , $R_{1-tt}^{\mathcal{N}}(\text{NP})$ —has Turing-complete sets but has no many-one complete sets. In fact, we show that in the same relativized world this class has 2-truth-table complete sets but lacks 1-truth-table complete sets. As part of the groundwork for our result, we prove that $R_{1-tt}^{\mathcal{N}}(\text{NP})$ has many equivalent forms having to do with ordered and parallel access to NP and $\text{NP} \cap \text{coNP}$.

1 Introduction

In this paper, we ask whether there are natural complexity classes for which the *existence* of many-one and Turing-complete sets can be distinguished. Many standard complexity classes—e.g., R , BPP , UP , FewP , $\text{NP} \cap \text{coNP}$ —are known that in some relativized worlds lack many-one complete (m-complete) sets, and that in some relativized worlds lack Turing-complete (T-complete) sets. However, for none of the classes just mentioned is there known

*Supported in part by grant NSF-INT-9513368/DAAD-315-PRO-fo-ab. Work done in part while visiting Friedrich-Schiller-Universität Jena. Email: edith@bamboo.lemoyne.edu.

†Supported in part by grants NSF-CCR-9322513 and NSF-INT-9513368/DAAD-315-PRO-fo-ab. Work done in part while visiting Friedrich-Schiller-Universität Jena. Email: lane@cs.rochester.edu.

‡Supported in part by grant NSF-INT-9513368/DAAD-315-PRO-fo-ab. Work done in part while visiting Le Moyne College. Email: hempel@informatik.uni-jena.de.

any relativized world in which the class (simultaneously) has T-complete sets but lacks m-complete sets. In fact, for $\text{NP} \cap \text{coNP}$ and BPP, Gurevich [Gur83] and Ambos-Spies [Amb86] respectively have shown that no such world can exist. In this paper, we will show that there is a downward closure of NP, $\text{R}_{1\text{-tt}}^{\text{SN}}(\text{NP})$, that in some relativized worlds simultaneously has T-complete sets and lacks m-complete sets.

In fact, $\text{R}_{1\text{-tt}}^{\text{SN}}(\text{NP})$ has even stronger properties. We will see that it *robustly*—i.e., in all relativized worlds, including the real world—has 2-truth-table complete (2-tt-complete) sets. Yet we will see that in our relativized world it lacks 1-tt-complete sets. Thus, this class displays a very crisp borderline between those reduction types under which it robustly has complete sets, and those reduction types under which it does not robustly have complete sets.

We now turn in more detail to describing what is currently known in the literature regarding robust completeness. Sipser [Sip82] first studied this notion, and showed that $\text{NP} \cap \text{coNP}$ and random polynomial time (R) do not robustly have m-complete sets. However, as alluded to in the first paragraph, Gurevich [Gur83] proved that, in each relativized world, $\text{NP} \cap \text{coNP}$ has m-complete sets if and only if $\text{NP} \cap \text{coNP}$ has T-complete sets. Thus, $\text{NP} \cap \text{coNP}$ cannot distinguish robust m-completeness from robust T-completeness. Ambos-Spies [Amb86] extended this by showing that no class closed downwards under Turing reductions can distinguish robust m-completeness from robust T-completeness.

Thus, the only candidates for distinguishing robust m-completeness from robust T-completeness within PSPACE are those classes in PSPACE that may lack m-complete sets yet that seem not to be closed downwards under Turing reductions. The classes R, UP, and FewP have been shown to potentially be of this form (see, respectively, [Sip82], [HH88], and [HJV93] for proofs that these classes do not robustly have m-complete sets¹). Unfortunately, these classes are also known to not robustly have T-complete sets [HJV93], and so these classes fail to distinguish robust m-completeness from robust T-completeness.

In fact, to the best of our knowledge, the literature contains only one type of class that distinguishes robust m-completeness from robust T-completeness—and that type is deeply unsatisfying. The type is certain “union” classes—namely, certain classes that either union incomparable classes or that union certain infinite hierarchies of bounded-access classes. Both exploit the fact that if such classes have some m-complete set it must fall into some

¹The study of robust completeness has been pursued in many papers. Of particular interest is the elegant work of Bovet, Crescenzi, and Silvestri [BCS92], which abstracts the issue of m-completeness away from particular classes via general conditions. Also, the other method of proving such results has been reasserted, in a very abstract and algebraic form, in the recent thesis of Borchert [Bor94], which re-poses abstractly the proof approach that was pioneered by Sipser ([Sip82], see also [Reg89]). Like the Bovet/Crescenzi/Silvestri approach, this method abstracts away from directly addressing completeness, in the case of this approach via characterizing completeness in terms of the issue of the existence of certain index sets (in the Borchert version, the discussion is abstracted one level further than this). In Section 4 we follow the Sipser/Regan/Borchert “index sets/enumeration” approach, in its non-algebraic formulation.

particular element of the union. An example of the “incomparable” case is that if $\text{NP} \cup \text{coNP}$ has m-complete sets then $\text{NP} = \text{coNP}$ (and $\text{NP} = \text{coNP}$ is not robustly true [BGS75]). An example (from [HJV93]) of the “infinite union of bounded-access classes” case is the boolean hierarchy [CGH⁺88], i.e.,

$$\text{BH} = \{L \mid L \leq_{btt}^p \text{SAT}\}.$$

From its definition, it is clear that SAT is T-complete (indeed, even bounded-truth-table complete) for BH. However, if BH had an m-complete set then that set (since it would be in BH) would have to be computable via some k -truth-table reduction to SAT, so there would be a \hat{k} such that $\text{BH} = \{L \mid L \leq_{\hat{k}-tt}^p \text{SAT}\}$, but this is known to not be robustly true [CGH⁺88].

We at this point mention an interesting related topic that this paper is not about, and with which our work should not be confused. That topic, in contrast to our attempt to distinguish *the existence of* m-complete and T-complete sets for a class, is the study of whether one can merely distinguish *the set of* m-complete and T-complete sets for a class. For example, various conditions (most strikingly, NP does not “have p-measure 0” [LM96]) are known such that their truth would imply that the class of NP-m-complete sets differs from the class of NP-T-complete sets. However, this does not answer our question, as NP robustly has m-complete sets and robustly has T-complete sets. The exact same comment applies to the work of Watanabe and Tang [WT92] that shows certain conditions under which the class of PSPACE-m-complete sets differs from the class of PSPACE-T-complete sets. Also of interest, but not directly related to our interest in the *existence* of complete sets, is the work of Longpré and Young [LY90] showing that within NP Turing reductions can be polynomially “faster” than many-one reductions.

As mentioned at the start of this section, in this paper we prove that $\text{R}_{1-tt}^{\text{SN}}(\text{NP})$ robustly has T-complete sets but does not robustly have m-complete sets. We actually prove the stronger result that $\text{R}_{1-tt}^{\text{SN}}(\text{NP})$ distinguishes robust 1-tt-completeness from robust 2-tt-completeness. This of course implies that there is a relativized world in which $\text{R}_{1-tt}^{\text{SN}}(\text{NP})$ has T-complete (even 2-tt-complete) sets but lacks m-complete (even 1-tt-complete) sets. It is important to note that this is not analogous to the “union” examples given two paragraphs ago. $\text{R}_{1-tt}^{\text{SN}}(\text{NP})$ is not a “union” class. Also, the mere fact that a class is defined in terms of some type of access to NP is not, in and of itself, enough to preclude robust m-completeness, as should be clear from the fact that $\text{R}_{1-tt}^p(\text{NP})$ and $\text{R}_{2-tt}^p(\text{NP})$ robustly have m-complete sets (note: $\text{R}_{1-tt}^p(\text{NP}) \subseteq \text{R}_{1-tt}^{\text{SN}}(\text{NP}) \subseteq \text{R}_{2-tt}^p(\text{NP})$).

Regarding the background of the reducibility \leq_{1-tt}^{SN} , we mention that Homer and Longpré ([HL94, Corollary 5], see also [OW91]) have recently proven that if any set that is \leq_m^p -hard for NP is \leq_{1-tt}^{SN} -reducible (or even \leq_{btt}^{SN} -reducible) to a sparse set then the polynomial hierarchy equals NP. Regarding the class $\text{R}_{1-tt}^{\text{SN}}(\text{NP})$, we consider $\text{R}_{1-tt}^{\text{SN}}(\text{NP})$ to be its most natural form. However, Section 3 proves that this class has

many equivalent characterizations (for example, it is exactly the class $P^{(NP \cap coNP, NP)}$ —what a P machine can compute via one $NP \cap coNP$ query made in parallel with one NP query). Section 3 also gives a candidate language for $R_{1-tt}^{SN}(NP)$ (namely PrimeSAT = $\{\langle i, F \rangle \mid i \in \text{PRIMES} \iff F \in \text{SAT}\}$) and notes that though $R_{1-tt}^{SN}(NP) \subseteq DP$,² the containment is strict unless the polynomial hierarchy collapses.

2 Preliminaries

For standard notions not defined here, we refer the reader to any computational complexity textbook, e.g., [BC93, Pap94, BDG95].

Unless otherwise stated or otherwise obvious from context, all strings will use the alphabet $\Sigma = \{0, 1\}$ and all sets will be collections of such strings. For every set A we will denote the characteristic function of A by χ_A . $A^{\leq k}$ denotes $\{x \mid x \in A \wedge |x| \leq k\}$. Strong nondeterministic reductions were introduced by Selman [Sel78] (with different nomenclature) and Long [Lon82]. The literature contains two potentially different notions of strong nondeterministic *truth-table* reducibility, one due to Long [Lon82] and the other due to Rich [Ric89] and Homer and Longpré [HL94]. (The notions differ, for example, regarding whether the query generation is single-valued or multivalued.) Throughout this paper, we use the notion of Homer and Longpré and Rich.

Definition 2.1 (see [Sel94b, SXB83]) *A function f is in $NPSV_t$ if there exists a nondeterministic polynomial-time Turing machine N such that, on each input x , it holds that*

1. *at least one computation path of $N(x)$ is an accepting path that outputs $f(x)$, and*
2. *every accepting computation path of $N(x)$ computes the same value, i.e., $f(x)$. (Note: rejecting computation paths are viewed as having no output.)*

Definition 2.2 ([HL94], see also [Ric89]) *For any constant k we say A is k -truth-table strong nondeterministic reducible to B ($A \leq_{k-tt}^{SN} B$) if there is a function in $NPSV_t$ that computes both (a) k strings x_1, x_2, \dots, x_k and (b) a predicate, α , of k boolean variables, such that x_1, x_2, \dots, x_k and α satisfy:*

$$x \in A \iff \alpha(\chi_B(x_1), \chi_B(x_2), \dots, \chi_B(x_k)).$$

Let \mathcal{C} be a complexity class. We say $A \leq_m^{p, \mathcal{C}[1]} B$ if and only if there is a function $f \in \text{FP}^{\mathcal{C}[1]}$ (i.e., computable via a deterministic polynomial-time Turing machine allowed one query to some oracle from \mathcal{C}) such that, for all x , $x \in A \iff f(x) \in B$.

²DP = $\{L \mid (\exists L_1, L_2 \in \text{NP})[L = L_1 - L_2]\}$ [PY84].

As is standard in the literature, for any strings of symbols a and b for which \leq_a^b is defined and any class \mathcal{C} , let $R_a^b(\mathcal{C}) = \{L \mid (\exists C \in \mathcal{C})[L \leq_a^b C]\}$.

Let $\langle \cdot, \cdot \rangle$ be any fixed pairing function with the standard nice properties (polynomial-time computability, polynomial-time invertibility).

We use DPTM (NPTM) as shorthand for “deterministic (nondeterministic) polynomial-time oracle Turing machine,” and we treat non-oracle Turing machines as oracle Turing machines that merely happen not to use their oracle tapes. Without loss of generality, we henceforward assume that DPTMs and NPTMs are clocked with clocks that are independent of the oracle. $M^A(x)$ denotes the computation of the DPTM M with oracle A on input x . At times, when the oracle is clear from context, we may write $M(x)$, omitting the oracle superscript(s) (such as $M^A(x)$).

Let $\{M_i\}$ and $\{N_i\}$ respectively be enumerations of deterministic and nondeterministic polynomial-time oracle Turing machines. Without loss of generality, let these enumerations be such that M_i and N_i run in (respectively, deterministic and nondeterministic) time $n^i + i$ and let them also be such that given i one can in polynomial time derive (as Turing machine code) M_i and N_i .

Definition 2.3 *Let \mathcal{C} and \mathcal{D} be complexity classes.*

1. [HHW] Let $M^{A:B}$ denote a DPTM M making one query to oracle A followed by one query to oracle B .³ Let

$$P^{\mathcal{C}:\mathcal{D}} = \{L \mid (\exists C \in \mathcal{C})(\exists D \in \mathcal{D})(\exists \text{ DPTM } M)[L = L(M^{C:D})]\}$$

2. [HHH97a] Let $M^{(A,B)}$ denote a DPTM M making, simultaneously, one query to oracle A and one query to oracle B . Let

$$P^{(\mathcal{C},\mathcal{D})} = \{L \mid (\exists C \in \mathcal{C})(\exists D \in \mathcal{D})(\exists \text{ DPTM } M)[L = L(M^{(C,D)})]\}.$$

Classes of the form $P^{\mathcal{C}:\mathcal{D}}$ were introduced and studied by Hemaspaandra, Hempel, and Wechsung [HHW]. They focused on the case in which \mathcal{C} and \mathcal{D} are levels of the boolean hierarchy. The present authors [HHH97a] first studied the case in which \mathcal{C} and \mathcal{D} are levels of the polynomial hierarchy. These papers propose and study the effect of the *order* of database access on the power of database-accessing machines. That line of research has led recently to the counterintuitive downward collapse result that, for each $k \geq 2$,

³We do not describe the mechanics of having two oracles, as any natural approach will do in the contexts with which we are dealing. For example, oracle machines can all have one oracle tape, with the query to the first oracle being contained in the tape cells to the right of the origin and the query to the second oracle being contained in the tape cells to the left of the origin, and with only the appropriate half being erased after entering the distinguished state denoting a query to that half. Alternatively and perhaps more naturally, one can allow the oracle machine to have one oracle tape per oracle.

$\Sigma_k^p = \Pi_k^p \iff P^{\Sigma_k^p[1]} = P^{\Sigma_k^p[2]}$ ([HHH,BF96], see also [HHH97c]), and to a number of other interesting results [Wag97,BC97].

Part 2 of Definition 2.3 is somewhat related to work of Selivanov [Sel94a]. This fact, and the comments of the rest of this paragraph, were noted independently by an earlier version of the present paper [HHH97b] and by Klaus Wagner ([Wag97], see also [BC97]), whose observations are in a more general form (namely, applying to more than two sets and to more abstract classes). We now discuss the basic facts known about the relationship between the classes of Selivanov (for the case of “ Δ ”s of two sets; see Wagner [Wag97] for the case of more than two sets) and the classes discussed in this paper. Selivanov studied refinements of the polynomial hierarchy. Among the classes he considered, those closest to the classes we study in this paper are his classes

$$\Sigma_i^p \Delta \Sigma_j^p = \{L \mid (\exists A \in \Sigma_i^p)(\exists B \in \Sigma_j^p)[L = A \Delta B]\},$$

where $A \Delta B = (A - B) \cup (B - A)$. Note, however, that his classes seem to be different from our classes. This can be immediately seen from the fact that all our classes are closed under complementation, but the main theorem of Selivanov ([Sel94a], see also the discussion and strengthening in [HHH97c]) states that no class of the form $\Sigma_i^p \Delta \Sigma_j^p$, with $i > 0$ and $j > 0$, is closed under complementation unless the polynomial hierarchy collapses. Nonetheless, the class $\Sigma_i^p \Delta \Sigma_j^p$ is not too much weaker than $P^{(\Sigma_i^p, \Sigma_j^p)}$, as it is not hard to see (by easy manipulations if $i \neq j$, and from the work of Wagner [Wag90] and Köbler, Schöning, and Wagner [KSW87] for the $i = j$ case) that, for all i and j , it holds that $\{L \mid (\exists L' \in \Sigma_i^p \Delta \Sigma_j^p)[L \leq_{1\text{-tt}}^p L']\} = P^{(\Sigma_i^p, \Sigma_j^p)}$.

3 Equivalent forms of $R_{1\text{-}tt}^{\mathcal{SN}}(\text{NP})$

In this section, we consider the class $R_{1\text{-}tt}^{\mathcal{SN}}(\text{NP})$ and note that this class is quite oblivious to definitional variations; it has many equivalent forms.

The following lemma is from [HHH97a] and will be useful here.⁴

Lemma 3.1 [HHH97a] *If \mathcal{C}_1 and \mathcal{C}_2 are classes such that \mathcal{C}_1 is closed downwards under $\leq_m^{p, \mathcal{C}_2[1]}$ then*

$$P^{\mathcal{C}_1:\mathcal{C}_2} = P^{\mathcal{C}_2:\mathcal{C}_1} = P^{(\mathcal{C}_1, \mathcal{C}_2)}.$$

Now we are prepared to state and prove the main theorem of this section, Theorem 3.2. It will follow easily from this theorem that $R_{1\text{-}tt}^{\mathcal{SN}}(\text{NP})$ is equivalent to ordered access to NP and $\text{NP} \cap \text{coNP}$, and also to parallel access to NP and $\text{NP} \cap \text{coNP}$ —with one query to each of NP and $\text{NP} \cap \text{coNP}$ allowed in each case. Theorem 3.2’s proof uses the following technique.

⁴The theorem, its asymmetry notwithstanding, is not mistyped. One does not need to additionally assume that \mathcal{C}_2 is closed downwards under $\leq_m^{p, \mathcal{C}_1[1]}$.

The theorem deals with $R_{1-tt}^{\mathcal{N}}(\mathcal{C})$, i.e., with a certain type of 1-truth-table reduction. A 1-truth-table has two bits of information—what to do (accept versus reject) if the answer is yes, and what to do (accept versus reject) if the answer is no—contained in the truth-table itself. Additionally, in $R_{1-tt}^{\mathcal{N}}(\mathcal{C})$ there is information in the (yes/no) answer from the \mathcal{C} query. The key trick in the proof (this occurs in the proof that $R_{1-tt}^{\mathcal{N}}(\mathcal{C}) \subseteq P^{(NP \cap \text{coNP}, \mathcal{C})}$) is to restructure this so that the effect of the 1-truth-table reduction to a \mathcal{C} query is simulated by one query each to $NP \cap \text{coNP}$ and \mathcal{C} . In effect, the $NP \cap \text{coNP}$ returns, in its one-bit answer, enough information about the two-bit truth-table that the base machine, working hand-in-hand with the \mathcal{C} query, can make do with the one bit rather than two.

Theorem 3.2 *For every class \mathcal{C} that is closed downwards under $\leq_m^{p, NP \cap \text{coNP}[1]}$ we have*

$$R_{1-tt}^{\mathcal{N}}(\mathcal{C}) = P^{NP \cap \text{coNP} : \mathcal{C}} = P^{\mathcal{C} : NP \cap \text{coNP}} = P^{(NP \cap \text{coNP}, \mathcal{C})}.$$

Proof: Note that the rightmost two equalities follow immediately from Lemma 3.1.

It remains to show that $R_{1-tt}^{\mathcal{N}}(\mathcal{C}) = P^{NP \cap \text{coNP} : \mathcal{C}}$. Let us first show the inclusion from right to left. So suppose $L \in P^{NP \cap \text{coNP} : \mathcal{C}}$, as witnessed by DPTM M , $A \in NP \cap \text{coNP}$, and $B \in \mathcal{C}$. Without loss of generality, assume that M always makes exactly one query to each of its oracles. According to the definition of $\leq_{1-tt}^{\mathcal{N}}$ reductions as given in Section 2, we have to find a set $C \in \mathcal{C}$ and a function $f \in \text{NPSV}_t$ computing a string y ($= y(x)$) and a predicate α ($= \alpha(x)$) such that

$$x \in L \iff \alpha(\chi_C(y)).$$

We specify C by setting C to equal B . Let y be the query that is asked by $M^{A:B}(x)$ to B when the first query is answered correctly, and let α be a predicate defined by

$$\alpha(i) \iff \left(i = 0 \wedge x \in L(M^{A:\emptyset}) \right) \vee \left(i = 1 \wedge x \in L(M^{A:\Sigma^*}) \right).$$

Set $f(x) = \langle y, \alpha \rangle$ and note that $f \in \text{NPSV}_t$ and also $x \in L \iff \alpha(\chi_B(y))$. This proves $P^{NP \cap \text{coNP} : \mathcal{C}} \subseteq R_{1-tt}^{\mathcal{N}}(\mathcal{C})$.

The proof will be completed if we can prove $R_{1-tt}^{\mathcal{N}}(\mathcal{C}) \subseteq P^{(NP \cap \text{coNP}, \mathcal{C})}$; let us do so. Let $L \in R_{1-tt}^{\mathcal{N}}(\mathcal{C})$. Let $B \in \mathcal{C}$ and NPSV_t function f witness (in the sense of Definition 2.2) that $L \in R_{1-tt}^{\mathcal{N}}(\mathcal{C})$. So say $f(x) = \langle \langle X_1(x), X_2(x) \rangle, z(x) \rangle$. In particular, let $(X_1(x), X_2(x))$ — $X_1(x), X_2(x) \in \{A, R\}$ where A stands for accept and R for reject—denote the truth-table that is output as the first component of $f(x)$, i.e., $X_1(x)$ denotes the behavior (accept or reject) that occurs if the answer to the query to \mathcal{C} is “no” and $X_2(x)$ denotes the behavior (accept or reject) that occurs if the answer to the query to \mathcal{C} is “yes.”

We define sets $E \in NP \cap \text{coNP}$ and $F \in R_m^{p, NP \cap \text{coNP}[1]}(\mathcal{C})$, and describe a DPTM \widehat{M} such that $L = L(\widehat{M}^{(E, F)})$. Let

$$E = \{x \mid (X_1(x), X_2(x)) \neq (A, R)\}$$

and

$$F = \{x \mid (X_1(x), X_2(x)) = (A, A) \vee ((X_1(x), X_2(x)) = (R, A) \wedge z(x) \in B) \vee ((X_1(x), X_2(x)) = (A, R) \wedge z(x) \in B)\}.$$

Note that $E \in \text{NP} \cap \text{coNP}$ and $F \in \mathcal{C}$ by our hypothesis that $\mathcal{C} \supseteq R_m^{p, \text{NP} \cap \text{coNP}[1]}(\mathcal{C})$. Furthermore, let $\widehat{M}^{(E, F)}$ on input x query “ $x \in E?$ ” and “ $x \in F?$ ” and accept if and only if either both queries are answered “yes” or both are answered “no.”

Note that $L = L(\widehat{M}^{(E, F)})$ —as can easily be seen by considering each of the four cases (R, R) , (R, A) , (A, R) , and (A, A) . Thus $R_{1-tt}^{\mathcal{N}}(\mathcal{C}) \subseteq P^{(\text{NP} \cap \text{coNP}, \mathcal{C})}$, which completes the proof of the theorem. \blacksquare

For any classes \mathcal{C}_1 and \mathcal{C}_2 , let $\mathcal{C}_1 \ominus \mathcal{C}_2 =_{\text{def}} \{L \mid (\exists A \in \mathcal{C}_1)(\exists B \in \mathcal{C}_2)[L = A - B]\}$.

Theorem 3.3 *For every class \mathcal{C} that is closed downwards under $\leq_m^{p, \text{NP} \cap \text{coNP}[1]}$, it holds that*

$$P^{\mathcal{C}[1]} \subseteq R_{1-tt}^{\mathcal{N}}(\mathcal{C}) \subseteq \mathcal{C} \ominus \mathcal{C}.$$

Proof: Note that by Theorem 3.2 we have $R_{1-tt}^{\mathcal{N}}(\mathcal{C}) = P^{(\text{NP} \cap \text{coNP}, \mathcal{C})}$. $P^{\mathcal{C}[1]} \subseteq R_{1-tt}^{\mathcal{N}}(\mathcal{C})$ follows immediately. For the second inclusion, namely $R_{1-tt}^{\mathcal{N}}(\mathcal{C}) \subseteq \mathcal{C} \ominus \mathcal{C}$, suppose $L \in R_{1-tt}^{\mathcal{N}}(\mathcal{C})$ and thus, by Theorem 3.2, $L \in P^{(\text{NP} \cap \text{coNP}, \mathcal{C})}$. Let $L \in P^{(\text{NP} \cap \text{coNP}, \mathcal{C})}$ be witnessed by DPTM M , $A \in \text{NP} \cap \text{coNP}$, and $B \in \mathcal{C}$. Without loss of generality assume that M makes, on every input, exactly one query to A and exactly one query to B . We describe two sets F_1 and F_2 , both from \mathcal{C} , such that $L = F_1 - F_2$. Before we come to the actual definition of F_1 and F_2 we introduce some notations that are similar to the ones used in the proof of Theorem 3.2. Let $(X_1(x), X_2(x))$ — $X_1(x), X_2(x) \in \{A, R\}$, where A stands for accept and R for reject—denote the 1-variable truth-table with respect to the query to B of $M^{(A, B)}(x)$ (given that the first query is answered correctly). That is,

$X_1(x)$ is the outcome of $M^{(A, \emptyset)}(x)$, and

$X_2(x)$ is the outcome of $M^{(A, \Sigma^*)}(x)$.

Let $q_2(x)$ denote the query asked to B by $M^{(A, B)}(x)$.

$$F_1 = \{x \mid (X_1(x), X_2(x)) = (A, A) \vee (X_1(x), X_2(x)) = (A, R) \vee ((X_1(x), X_2(x)) = (R, A) \wedge q_2(x) \in B)\}$$

and

$$F_2 = \{x \mid (X_1(x), X_2(x)) = (A, R) \wedge q_2(x) \in B\}.$$

Note that both F_1 and F_2 are indeed in \mathcal{C} , since both sets are clearly in $R_m^{p, \text{NP} \cap \text{coNP}[1]}(\mathcal{C})$ and we know by assumption that this class equals \mathcal{C} . On the other hand one can easily verify that $L = F_1 - F_2$ and thus $L \in \mathcal{C} \ominus \mathcal{C}$. \blacksquare

Let us apply to $R_{1-tt}^{\mathcal{N}}(\text{NP})$ the results just obtained. The following well-known fact will be helpful.

Lemma 3.4 $R_m^{p, \text{NP} \cap \text{coNP}[1]}(\text{NP}) = \text{NP}$.

Proof: The inclusion from right to left is clear. Consider the inverse inclusion and note that clearly $R_m^{p, \text{NP} \cap \text{coNP}[1]}(\text{NP}) \subseteq \text{NP}^{\text{NP} \cap \text{coNP}[1]}$. However, $\text{NP} = \text{NP}^{\text{NP} \cap \text{coNP}}$ (first obtained in [Lon82]—which never states this but does subtly use it) and thus $R_m^{p, \text{NP} \cap \text{coNP}[1]}(\text{NP}) = \text{NP}$. \blacksquare

From Lemma 3.4 and Theorems 3.2 and 3.3 we have the following two corollaries for $R_{1-tt}^{\mathcal{N}}(\text{NP})$.

Corollary 3.5 $R_{1-tt}^{\mathcal{N}}(\text{NP}) = \text{P}^{\text{NP} \cap \text{coNP} : \text{NP}} = \text{P}^{\text{NP} : \text{NP} \cap \text{coNP}} = \text{P}^{(\text{NP} \cap \text{coNP}, \text{NP})}$.

Corollary 3.6 $\text{P}^{\text{NP}[1]} \subseteq R_{1-tt}^{\mathcal{N}}(\text{NP}) \subseteq \text{DP}$.

Since $R_{1-tt}^{\mathcal{N}}(\text{NP})$ is closed under complementation but DP is suspected not to be, the second inclusion probably is strict (we note in passing that, due to the closure under complementation of $R_{1-tt}^{\mathcal{N}}(\text{NP})$, $R_{1-tt}^{\mathcal{N}}(\text{NP}) \subseteq \text{DP} \iff R_{1-tt}^{\mathcal{N}}(\text{NP}) \subseteq \text{DP} \cap \text{coDP}$).

Corollary 3.7 *If $R_{1-tt}^{\mathcal{N}}(\text{NP}) = \text{DP}$ (equivalently, if $R_{1-tt}^{\mathcal{N}}(\text{NP}) = \text{DP} \cap \text{coDP}$) then the boolean hierarchy collapses (and thus, by [Kad88], the polynomial hierarchy also collapses).*

Though Corollary 3.7 gives strong evidence that the second inclusion of Corollary 3.6 is strict, we know of no class collapse that follows from the assumption that the first inclusion is not strict (though it is easy to directly construct an oracle relative to which the first inclusion is strict, and clearly the first inclusion must be strict in the relativized world we are going to construct in Section 4 in which $R_{1-tt}^{\mathcal{N}}(\text{NP})$ lacks m-complete sets). Can one prove that $\text{P}^{\text{NP}[1]} = R_{1-tt}^{\mathcal{N}}(\text{NP})$ implies some surprising collapse of complexity classes?

What types of sets are in $R_{1-tt}^{\mathcal{N}}(\text{NP})$? Define $\text{PrimeSAT} = \{\langle i, f \rangle \mid i \in \text{PRIMES} \iff f \in \text{SAT}\}$. Clearly $\text{PrimeSAT} \in \text{P}^{(\text{NP} \cap \text{coNP}, \text{NP})}$ and thus, by Corollary 3.5, $\text{PrimeSAT} \in R_{1-tt}^{\mathcal{N}}(\text{NP})$. On the other hand, $\text{PrimeSAT} \in \text{P}^{(\text{ZPP} \cap \text{UP} \cap \text{coUP}, \text{NP})}$ (since $\text{PRIMES} \in \text{ZPP} \cap \text{UP} \cap \text{coUP}$ [AH87, FK92]), so it seems somewhat unlikely that PrimeSAT is m-complete for $R_{1-tt}^{\mathcal{N}}(\text{NP})$. In fact, though (see the discussion in Section 4) $R_{1-tt}^{\mathcal{N}}(\text{NP})$ robustly has 2-tt-complete sets, nonetheless $R_{1-tt}^{\mathcal{N}}(\text{NP})$ may well lack 1-tt-complete sets. In fact, we will in the next section construct a relativized world in which $R_{1-tt}^{\mathcal{N}}(\text{NP})$ has no 1-tt-complete set.

Finally, we note that $\text{P}^{\text{NP} \cap \text{coNP} : \text{NP}} = R_{1-tt}^{\mathcal{N}}(\text{NP})$ is a case where guarded database (oracle) access seems more powerful than standard access. So-called guarded reductions were introduced by Grollmann and Selman [GS88] (there called “smart reductions”), and were further investigated by Cai, Hemaspaandra (then Hemachandra), and Vyskoč [CHV93]. In light of Corollary 3.5 we will now look at guarded oracle access to $\text{NP} \cap \text{coNP}$ in the

context of Definition 2.3, in order to see whether guarded access yields yet another equivalent form of $R_{1-tt}^{\mathcal{N}}(\text{NP})$. We will see that it seems not to.

Let $\text{P}^{\text{NP}:\{\text{NP} \cap \text{coNP}\}}$ be the class of languages that are recognized by some DPTM M that makes two sequential queries, the first to some NP set A , and the second to some NP set B , and such that it also holds that there is another NP set C such that, for all y :

if there is an x such that $M^{A:B}(x)$ given the correct answer to its query to A (if any) queries “ $y \in B$?” then: $y \in B \iff y \notin C$.

In other words, M is allowed an ordinary query to NP, followed by a query that must be “NP \cap coNP-like.” However, this is not necessarily the same as allowing an NP \cap coNP query (indeed, see Theorem 3.8). The key point is that on strings never asked by $M^{A:B}$ to B , B and C need not be complementary. Informally, M and A guard B against queries where B might fail to complement C .⁵

Theorem 3.8 $R_{1-tt}^{\mathcal{N}}(\text{NP}) = \text{P}^{\text{NP}:\text{NP} \cap \text{coNP}} \subseteq \text{DP} \cap \text{coDP} \subseteq \text{P}^{\text{NP}:\{\text{NP} \cap \text{coNP}\}}$.

Proof: The first equality is part of Corollary 3.5. The inclusion $R_{1-tt}^{\mathcal{N}}(\text{NP}) \subseteq \text{DP} \cap \text{coDP}$ follows immediately from Corollary 3.6 as noted after that corollary.

It remains to show $\text{DP} \cap \text{coDP} \subseteq \text{P}^{\text{NP}:\{\text{NP} \cap \text{coNP}\}}$. Let $L \in \text{DP} \cap \text{coDP}$. So there exist NP sets L_1, L_2, L_3 , and L_4 such that $L = L_1 - L_2 = \overline{L_3 - L_4}$. Following Cai et al. [CGH⁺88], we can without loss of generality assume that $L_2 \subseteq L_1$ and $L_4 \subseteq L_3$.

We make the following observations:

1. $(x \notin L_2 \wedge x \notin L_4) \Rightarrow (x \in L_1 \iff x \notin L_3)$, and
2. $(x \in L_2 \vee x \in L_4) \Rightarrow (x \in L_4 \iff x \notin L_2)$.

Define $E = \{x \mid x \in L_2 \vee x \in L_4\}$ and $F = \{\langle x, 0 \rangle \mid x \in L_1\} \cup \{\langle x, 1 \rangle \mid x \in L_4\}$, and note that clearly E and F are in NP.

Let M be a DPTM that on input x first queries “ $x \in E$?” and if it gets the answer i , where $i \in \{0, 1\}$ and 0 stands for “no” and 1 stands for “yes,” queries “ $\langle x, i \rangle \in F$?” and accepts if and only if this second query is answered “yes.”

By our above observations we know that M “smartly” accesses F and hence $L \in \text{P}^{\text{NP}:\{\text{NP} \cap \text{coNP}\}}$. ■

4 Completeness

In this section we prove that there is a relativized world in which $R_{1-tt}^{\mathcal{N}}(\text{NP})$ has no 1-tt-complete sets (and thus no m-complete sets). We will note that $R_{1-tt}^{\mathcal{N}}(\text{NP})$ robustly has

⁵There is no need to similarly define $\text{P}^{\{\text{NP} \cap \text{coNP}\}:\text{NP}}$ and $\text{P}^{\{\{\text{NP} \cap \text{coNP}\}, \text{NP}\}}$, as it is clear that $R_{1-tt}^{\mathcal{N}}(\text{NP}) = \text{P}^{\{\text{NP} \cap \text{coNP}\}:\text{NP}} = \text{P}^{\{\{\text{NP} \cap \text{coNP}\}, \text{NP}\}}$. This is just a reflection of the known fact that, in most contexts, if the “first” query is guarded it can as well be unguarded as the very fact that it is asked is a certificate that the query obeys the appropriate promise [CHV93].

2-tt-complete sets. Thus we show even more, namely that $R_{1-tt}^{\mathcal{N}}(\text{NP})$ distinguishes robust 1-tt-completeness from robust 2-tt-completeness (and thus it also distinguishes robust m-completeness from robust T-completeness).

To discuss relativized completeness we must define relativized reductions and the natural relativizations of our classes. So that our theorems are fair, we choose full relativizations (see [Rog67]), i.e., relativizations in which both the reductions and the classes may access the oracle. However, as Theorem 4.2 will show, many different statements regarding completeness—some involving partial relativizations—are equivalent. In fact, we will make use of some of these equivalences in proving our result.

Definition 4.1 1. Let $\leq_{1-tt}^{\mathcal{N},A}$ be as in Definition 2.2, except with NPSV_t replaced by NPSV_t^A .

2. (Full relativization of $R_{1-tt}^{\mathcal{N}}(\text{NP})$) $(R_{1-tt}^{\mathcal{N}}(\text{NP}))^A =_{\text{def}} R_{1-tt}^{\mathcal{N},A}(\text{NP}^A)$, i.e., $\{L \mid (\exists C \in \text{NP}^A)[L \leq_{1-tt}^{\mathcal{N},A} C]\}$.
3. Let $\leq_{1-tt}^{p,A}$ (respectively, \leq_{1-tt}^p [LLS75]) be as in Part 1 of the present definition (respectively, as in Definition 2.2) except with NPSV_t^A (respectively, NPSV_t) replaced by FP_t^A (respectively, FP_t), where FP_t denotes the deterministic polynomial-time computable functions.⁶ Many-one and 2-truth-table reductions are relativized in the obvious analogous ways.
4. (Full relativization of $P^{(\text{NP} \cap \text{coNP}, \text{NP})}$) $(P^{(\text{NP} \cap \text{coNP}, \text{NP})})^A =_{\text{def}} \{L \mid L \text{ is recognized by a deterministic polynomial-time Turing machine that makes, in parallel, at most one query to } \text{NP}^A \cap \text{coNP}^A \text{ and at most one query to } \text{NP}^A, \text{ and that additionally has—before the parallel round or after the parallel round or both—unlimited access to } A\}$.

Theorem 4.2 Let A be any set. All of the following twelve statements are equivalent.

1. $(R_{1-tt}^{\mathcal{N}}(\text{NP}))^A$ has $\leq_{1-tt}^{p,A}$ -complete sets.
2. $(R_{1-tt}^{\mathcal{N}}(\text{NP}))^A$ has \leq_{1-tt}^p -complete sets.
3. $(R_{1-tt}^{\mathcal{N}}(\text{NP}))^A$ has $\leq_m^{p,A}$ -complete sets.
4. $(R_{1-tt}^{\mathcal{N}}(\text{NP}))^A$ has \leq_m^p -complete sets.
- 5–8. The same as Parts 1–4, with $(R_{1-tt}^{\mathcal{N}}(\text{NP}))^A$ replaced by $(P^{(\text{NP} \cap \text{coNP}, \text{NP})})^A$.
- 9–12. The same as Parts 1–4, with $(R_{1-tt}^{\mathcal{N}}(\text{NP}))^A$ replaced by $P^{(\text{NP}^A \cap \text{coNP}^A, \text{NP}^A)}$.

⁶This definition is equivalent to the more traditional “generator plus evaluator” 1-truth-table definition with, as is natural, both the generator and the evaluator relativized.

Theorem 4.2 follows from Lemma 4.3, Lemma 4.4, and the fact that $(R_{1-tt}^{SN}(\text{NP}))^A$ is clearly closed downwards under $\leq_{1-tt}^{p,A}$ reductions.

Lemma 4.3 *For each set A , $(R_{1-tt}^{SN}(\text{NP}))^A = (P^{(\text{NP} \cap \text{coNP}, \text{NP})})^A = P^{(\text{NP}^A \cap \text{coNP}^A, \text{NP}^A)}$.*

Lemma 4.3 is essentially a relativized version of part of Corollary 3.5, plus the observation that the technique used in the second half of the proof of that result (that is, the second half of the proof of Theorem 3.2, in the case $\mathcal{C} = \text{NP}$) in fact can easily show not just $(R_{1-tt}^{SN}(\text{NP}))^A \subseteq (P^{(\text{NP} \cap \text{coNP}, \text{NP})})^A$, but even $(R_{1-tt}^{SN}(\text{NP}))^A \subseteq P^{(\text{NP}^A \cap \text{coNP}^A, \text{NP}^A)}$.

Most non-completeness proofs of the Sipser/Regan/Borchert school (i.e., proofs based on tainting enumerations) use a bridge between the existence of \mathcal{C}^A - $\leq_m^{p,A}$ -complete sets and the existence of \mathcal{C}^A - \leq_m^p -complete sets. Here, we extend that link to also embrace 1-truth-table reductions.

Lemma 4.4 *Let \mathcal{D} be any class (quite possibly a relativized class, such as $(R_{1-tt}^{SN}(\text{NP}))^A$) that is closed downwards under $\leq_{1-tt}^{p,A}$ reductions. Then the following four statements are equivalent: (a) \mathcal{D} has \leq_m^p -complete sets, (b) \mathcal{D} has $\leq_m^{p,A}$ -complete sets, (c) \mathcal{D} has \leq_{1-tt}^p -complete sets, (d) \mathcal{D} has $\leq_{1-tt}^{p,A}$ -complete sets.*

Proof: Since every \leq_m^p -complete set for \mathcal{D} is also $\leq_m^{p,A}$ -, \leq_{1-tt}^p -, and $\leq_{1-tt}^{p,A}$ -complete for \mathcal{D} , we have to show only that if \mathcal{D} is closed downwards under $\leq_{1-tt}^{p,A}$ reductions and \mathcal{D} has $\leq_{1-tt}^{p,A}$ -complete sets, then \mathcal{D} also has \leq_m^p -complete sets.

So let L be $\leq_{1-tt}^{p,A}$ -complete for \mathcal{D} . Define $L' = \{\langle x, i, 0^k \rangle \mid M_i^{L,A}(x), \text{ when run allowing at most one oracle query to } L \text{ during the run but allowed unlimited access to } A, \text{ accepts within } k \text{ steps}\}$. Note that $L' \leq_{1-tt}^{p,A} L$, and thus $L' \in \mathcal{D}$.

Also, L' is \leq_m^p -hard for \mathcal{D} . To verify this let $B \in \mathcal{D}$. Then by the $\leq_{1-tt}^{p,A}$ -completeness of L , $B \leq_{1-tt}^{p,A} L$, say via machine M_j . So $f(x) = \langle x, j, 0^{|x|^j+j} \rangle$ is a \leq_m^p reduction from B to L' . This proves that L' is \leq_m^p -complete for \mathcal{D} . \blacksquare

As is standard in the Sipser/Regan/Borchert approach to establishing non-completeness, we wish to characterize the existence of complete sets via the issue of the existence of a certain index set. Lemma 4.5 does this. Since it is quite similar to the analogous lemmas in previous non-completeness papers (see, e.g., [HH88, Lemmas 2.7 and 4.2]) we do not include the proof. We do, however, mention the following points. The lemma draws freely on Theorem 4.2. Also, the claim in Lemma 4.5 regarding P and P^A being equivalent (in that context) is an invocation of a trick from the literature [HH88, p. 134].

Lemma 4.5 *For every oracle A , $(R_{1-tt}^{SN}(\text{NP}))^A$ has $\leq_{1-tt}^{p,A}$ -complete sets if and only if there exists a P set (equivalently, a P^A set) I of index quadruples such that*

1. $(\forall \langle i, j, k, l \rangle)[\langle i, j, k, l \rangle \in I \Rightarrow L(N_j^A) = \overline{L(N_k^A)}], \text{ and}$

2. $P^{(NP^A \cap \text{coNP}^A, NP^A)} = \{L \left(M_i^{(L(N_j^A), L(N_l^A))} \right) \mid (\exists k)[\langle i, j, k, l \rangle \in I]\}$, and
3. $(\forall \langle i, j, k, l \rangle)[\langle i, j, k, l \rangle \in I \Rightarrow (\forall x)[\text{In the run of } M_i^{(L(N_j^A), L(N_l^A))}(x), M_i \text{ makes at most one round of truth-table queries and that round consists of at most one query (simultaneously) to each part of its } \left(L(N_j^A), L(N_l^A) \right) \text{ oracle}]]$.

We now prove our non-completeness claim.

Theorem 4.6 *There is a recursive oracle A such that $(R_{1\text{-tt}}^{SN}(\text{NP}))^A$ lacks $\leq_{1\text{-tt}}^{p,A}$ -complete sets.*

Proof: The proof consists of the construction of a recursive set A such that there exists no P set I having properties 1, 2, and 3 of Lemma 4.5.

Let $\{\widehat{M}_i\}$ be a standard enumeration of deterministic polynomial-time Turing machines. Let $\{N_i\}$ be the enumeration of nondeterministic polynomial-time oracle Turing machines described in Section 2. Let $\{M_i\}$ be an enumeration of deterministic polynomial-time oracle Turing machines satisfying all the properties of the enumeration $\{M_i\}$ described in Section 2 and having the additional property that every machine from $\{M_i\}$, on every input x , makes exactly one parallel round of exactly one query to each of its two oracles.

It is clear that there is such an enumeration $\{M_i\}$ and that this enumeration ensures that for each oracle A , for each i, j , and l , and for each input x , the queries asked by $M_i^{(L(N_j^A), L(N_l^A))}(x)$ to $L(N_j^A)$ and $L(N_l^A)$ are independent of A .

To show that there exists no P set I having properties 1, 2, and 3 of Lemma 4.5, we will diagonalize against all P machines in such a way that eventually for every P machine \widehat{M}_h (let h_e denote the e th string, $h_e = \langle i, j, k, l \rangle$, accepted by \widehat{M}_h) at least one of the following holds:

Goal 1 \widehat{M}_h accepts some string $h_e = \langle \widehat{i}, \widehat{j}, \widehat{k}, \widehat{l} \rangle$ such that $L(N_{\widehat{j}}^A) \neq \overline{L(N_{\widehat{k}}^A)}$. In this case, \widehat{M}_h does not accept a set of index quadruples I having property 1 of Lemma 4.5.

Goal 2 There exists a set $D_h \in P^{(NP^A \cap \text{coNP}^A, NP^A)}$ such that, for every $h_e = \langle i, j, k, l \rangle$ accepted by \widehat{M}_h , $D_h \neq L \left(M_i^{(L(N_j^A), L(N_l^A))} \right)$. (D_h will be explicitly defined later in the proof.) Thus \widehat{M}_h does not accept a set of index quadruples covering $P^{(NP^A \cap \text{coNP}^A, NP^A)}$ and hence \widehat{M}_h accepts a set not having property 2 of Lemma 4.5.

In the proof we will build a list, CAN , of canceled pairs (h, e) . We add a pair (h, e) , $h_e = \langle i, j, k, l \rangle$, to the list CAN when either Goal 1 has been met for h (in this case all (h, e') , $e' \in \mathbb{N}$, are marked canceled), or h_e is consistent with Goal 2 for h (i.e., with $h_e = \langle i, j, k, l \rangle$, $D_h \neq L \left(M_i^{(L(N_j^A), L(N_l^A))} \right)$). We describe the former case as a type 1 cancellation and the latter case as a type 2 cancellation.

Now let us define the languages D_h . For every $h \geq 1$, let

$$D_h = \{ 0^n \mid n \geq 3 \wedge (\exists k \geq 1)[n = (p_h)^k] \wedge [((\exists y)[|y| = n - 2 \wedge 00y \in A]) \iff ((\exists z)[|z| = n - 2 \wedge 11z \in A])]\},$$

where p_h denotes the h th prime.

We will always construct A so that, for all h such that for no \hat{e} is (h, \hat{e}) ever involved in a type 1 cancellation, it holds that: for each n such that, for some k , $n = (p_h)^k$:

$$\circledast \quad ((\exists y)[|y| = n - 2 \wedge 00y \in A]) \iff \neg((\exists y)[|y| = n - 2 \wedge 01y \in A]).$$

Note that \circledast will ensure that—for those h that are never involved in a type 1 cancellation—each such D_h will belong to $P^{(NP^A \cap \text{coNP}^A, NP^A)}$ (namely, as the right-hand side of the “ \iff ” of the definition of D_h is an NP^A type query, and the left-hand side of the “ \iff ” of the definition of D_h is in effect made into an $NP^A \cap \text{coNP}^A$ type query by \circledast).

Construction of $A =_{\text{def}} \bigcup_{m \geq 0} A_m$:

Stage 0: $CAN = \emptyset$, $A_0 = \emptyset$

Stage m , $m > 0$: Consider the uncanceled pair (h, e) , $e < m$, $(h, e) \notin CAN$, for which $h + e$ is smallest.

If

- (i) no such pair exists, or
- (ii) any length m string was queried at any previous stage,⁷ or
- (iii) m is not of the form $(p_h)^q$ for some $q \in \mathbb{N}$, where p_h is the h th prime, or
- (iv) the pair (h, e) , $h_e = \langle i, j, k, l \rangle$, that the above rule chooses (if any) is such that $3((m^i + i)^{\max(j, k, l)} + \max(j, k, l)) \geq 2^{m-16}$,⁸

then set $A_m = A_{m-1} \cup \{0^m\}$ (in order to maintain \circledast)⁹ and go to Stage $m + 1$.

⁷This condition is *never* satisfied. We include it just to emphasize that it is not an issue. The reason it is not an issue is discussed in Footnote 9.

⁸If $3((m^i + i)^{\max(j, k, l)} + \max(j, k, l)) < 2^{m-16}$ it (easily) holds that on the run of $M_i^{(L(N_j^A), L(N_l^A))}(0^m)$ the maximum number of queries to A made on any one path of $N_j^A(q_1)$ plus the maximum number of queries to A made on any one path of $N_k^A(q_1)$ plus the maximum number of queries to A made on any one path of $N_l^A(q_2)$ is less than 2^{m-16} , where q_1 and q_2 respectively denote the queries to $L(N_j^A)$ and $L(N_l^A)$ made by $M_i^{(L(N_j^A), L(N_l^A))}(0^m)$.

⁹Note that 0^m is free to put in, and so we do not have to search for some unqueried string chosen from $00\Sigma^{m-2}$. The reason we can use 0^m is that, as will soon become clear, each time a stage m' touches strings of length greater than m' , that stage then “jumps forward in time” (while maintaining appropriate codings in light of \circledast) to a stage m'' such that m'' is strictly greater than the length of any string queried at stage m' . In short, at the current point in the proof, *no* strings of length m have been queried or frozen, and thus in particular 0^m has never been queried or frozen.

Otherwise (i.e., if the “if” above is not satisfied), let (h, e) be the selected pair, $h_e = \langle i, j, k, l \rangle$. Define $\gamma = (m^i + i)^{\max(j, k, l)} + \max(j, k, l)$. Note that γ is an upper bound on the length of the strings in A that can be queried at this stage by any of M_i , N_j , N_k (when run on whatever query N_j is run on), and N_l . Let $Protectcodings_m$ denote the class of all sets E having exactly one string at each length \hat{i} , $m < \hat{i} \leq \gamma$, and such that E meets \circledast for each length \hat{i} , $m < \hat{i} \leq \gamma$, and such that E has no strings other than those just described. As noted in Footnote 9, no strings at lengths in this range have yet been queried, so we do not have to worry about already-frozen strings existing at these lengths. The reason we must include $Protectcodings_m$ in the condition that distinguishes between Case 1 and Case 2 is that one of the two possible ways the “ \neq ” of the Case 1 test can be satisfied (namely, if there is a string that is not in the set on the left-hand side and that is in the set on the right-hand side) requires us to freeze *rejecting* behavior of N_j and N_k . This involves freezing (i.e., fixing permanently the membership status regarding the oracle) an exponential number of strings—enough to ruin any attempts to satisfy \circledast at length $m + 1$ for example. Looking at $Protectcodings_m$ allows us to handle m and the problem m might cause at lengths $m + 1, m + 2, \dots, \gamma$ in an integrated fashion. In particular, simultaneously with choosing a length m extension we will choose an appropriate extension that handles the \circledast coding for lengths $m + 1, m + 2, \dots, \gamma$. There are two cases.

Case 1: For some $B \subseteq \Sigma^m$, for some $B' \in Protectcodings_m$,

$$\left(L \left(N_j^{A_{m-1} \cup B \cup B'} \right) \right)^{\leq m^i + i} \neq \left(\overline{L \left(N_k^{A_{m-1} \cup B \cup B'} \right)} \right)^{\leq m^i + i}.^{10}$$

In this case, we have a type 1 cancellation. So, for each $e' \in \mathbb{N}$, add (h, e') to CAN . For each $m \leq p \leq \gamma$, set $A_p = A_{m-1} \cup B \cup B'$. Go to stage $\gamma + 1$. Note that in this case we do not necessarily maintain \circledast at the current length in the construction of A . However, since we cancel “the entire machine M_h ” (by canceling all pairs (h, e')) and thus have successfully diagonalized against machine M_h , achieving Goal 1, we do not need in this case to argue regarding the set D_h and thus do not have to ensure that for this specific h , $D_h \in P^{(NP^A \cap coNP^A, NP^A)}$. We have, however, maintained \circledast at lengths $m + 1, m + 2, \dots, \gamma$.

Case 2: For every $B \subseteq \Sigma^m$, for every $B' \in Protectcodings_m$,

$$\left(L \left(N_j^{A_{m-1} \cup B \cup B'} \right) \right)^{\leq m^i + i} = \left(\overline{L \left(N_k^{A_{m-1} \cup B \cup B'} \right)} \right)^{\leq m^i + i}.$$

Note that this tells us that with respect to all oracles of the form $A_{m-1} \cup B \cup B'$, $B \subseteq \Sigma^m$ and $B' \in Protectcodings_m$, N_j and N_k are complementary at all lengths

¹⁰The $m^i + i$ bounds here are to ensure that the construction yields a recursive oracle.

that might be queried by $M_i^{(L(N_j^A), L(N_l^A))}(0^m)$. We are now going to exploit this fact in order to achieve a type 2 cancellation of the pair (h, e) . Note that we now must be careful in adding strings to our oracle set, since we have to maintain \circledast at length m . To satisfy \circledast at lengths $m + 1, m + 2, \dots, \gamma$, let B'' be any fixed member of $Protectcodings_m$; we will make B'' a part of the oracle extension.

Recall that the machine M_i makes one query round consisting of one query each (in parallel) to $L(N_j^A)$ and $L(N_l^A)$. We consider the action of

$$M_i^{(L(N_j^{A_{m-1} \cup B''}), L(N_l^{A_{m-1} \cup B''}))}(0^m).$$

There are four cases depending on the answers to the two queries $M_i^{(L(N_j^{A_{m-1} \cup B''}), L(N_l^{A_{m-1} \cup B''}))}(0^m)$ makes. We henceforward assume that the query to $L(N_j^{A_{m-1} \cup B''})$ is answered “yes.” The other case (“no”) is omitted as it is very analogous (note that in the case we are in, i.e., Case 2, if the answer to the query to $L(N_j^{A_{m-1} \cup B''})$ is “no” then the same query to $L(N_k^{A_{m-1} \cup B''})$ is answered “yes”).

Regarding the query to $L(N_l^{A_{m-1} \cup B''})$, the hard case is if it gets the answer “no,” as if it gets the answer “yes,” we freeze the lexicographically first accepting path (call it ς) and then perform a simpler version of the proof that is about to come.¹¹ So let us focus just on the case where $L(N_l^{A_{m-1} \cup B''})$ says “no.”

Freeze the lexicographically first accepting path ϱ of $N_j^{A_{m-1} \cup B''}$ on the input on which it is called. Let $guarded_m$ be all queries to A made on path ϱ .

Case 2a: There is some string, α , in $(00\Sigma^{m-2} \cup 01\Sigma^{m-2}) - guarded_m$ such that $N_l^{A_{m-1} \cup B'' \cup \{\alpha\}}$ accepts when run on the query asked to it by $M_i^{(L(N_j^{A_{m-1} \cup B'' \cup \{\alpha\}}), L(N_l^{A_{m-1} \cup B'' \cup \{\alpha\}}))}(0^m)$.

If either

- (a) $M_i^{(L(N_j^{A_{m-1} \cup B'' \cup \{\alpha\}}), L(N_l^{A_{m-1} \cup B'' \cup \{\alpha\}}))}(0^m)$ accepts and the first two bits of α are 00, or
- (b) $M_i^{(L(N_j^{A_{m-1} \cup B'' \cup \{\alpha\}}), L(N_l^{A_{m-1} \cup B'' \cup \{\alpha\}}))}(0^m)$ rejects and the first two bits of α are 01,

then we have achieved a type 2 cancellation for (h, e) , i.e., h_e codes a machine that does not accept D_h (actually, D_h defined with respect to A_m as we are about to define A_m , but the rest of the construction of A will not alter the achieved

¹¹Essentially one has to change the definition of $guarded_m$ (in order to include the queries made to A along ς) and do Case 2a with slight modifications.

cancellation). For each $m \leq p \leq \gamma$, set $A_p = A_{m-1} \cup B'' \cup \{\alpha\}$ and add (h, e) to CAN . Note that we have maintained \otimes at the current length, and at lengths $m+1, \dots, \gamma$. Go to stage $\gamma+1$.

Otherwise (i.e., if the “if” above is not satisfied) find a string z , $z \in 11\Sigma^{m-2}$, such that $z \notin \text{guarded}_m$ and z is not queried on the lexicographically first accepting path of $N_l^{A_{m-1} \cup B'' \cup \{\alpha\}}(q')$, where q' is the query asked to $N_l^{A_{m-1} \cup B'' \cup \{\alpha\}}$ by $M_i^{(L(N_j^{A_{m-1} \cup B'' \cup \{\alpha\}}), L(N_l^{A_{m-1} \cup B'' \cup \{\alpha\}}))}(0^m)$. Note that such a string z easily exists by (iv) (from beginning of stage m).

Note that the following two items hold:

1. Either
 - (a) $M_i^{(L(N_j^{A_{m-1} \cup B'' \cup \{\alpha\}}), L(N_l^{A_{m-1} \cup B'' \cup \{\alpha\}}))}(0^m)$ accepts and the first two bits of α are 01, or
 - (b) $M_i^{(L(N_j^{A_{m-1} \cup B'' \cup \{\alpha\}}), L(N_l^{A_{m-1} \cup B'' \cup \{\alpha\}}))}(0^m)$ rejects and the first two bits of α are 00.
2. z is not queried on the lexicographically first accepting path of $N_j^{A_{m-1} \cup B''}$ when run on the query asked to it by $M_i^{(L(N_j^{A_{m-1} \cup B'' \cup \{\alpha\}}), L(N_l^{A_{m-1} \cup B'' \cup \{\alpha\}}))}(0^m)$, and is not queried on the lexicographically first accepting path of N_l^A when run on the query asked by $M_i^{(L(N_j^{A_{m-1} \cup B'' \cup \{\alpha\}}), L(N_l^{A_{m-1} \cup B'' \cup \{\alpha\}}))}(0^m)$.

For each $m \leq p \leq \gamma$, set $A_p = A_{m-1} \cup B'' \cup \{\alpha\} \cup \{z\}$. We have achieved a type 2 cancellation of (h, e) , so we add (h, e) to CAN . Note that we have maintained \otimes at lengths $m, m+1, \dots, \gamma$. Go to stage $\gamma+1$.

Case 2b: For each string α in $(00\Sigma^{m-2} \cup 01\Sigma^{m-2}) - \text{guarded}_m$ it holds that $N_l^{A_{m-1} \cup B'' \cup \{\alpha\}}$ rejects when run on the query asked to it by $M_i^{(L(N_j^{A_{m-1} \cup B'' \cup \{\alpha\}}), L(N_l^{A_{m-1} \cup B'' \cup \{\alpha\}}))}(0^m)$.

If $M_i^{(L(N_j^{A_{m-1} \cup B''}), L(N_l^{A_{m-1} \cup B''}))}(0^m)$ accepts then let \hat{w} be any string in $00\Sigma^{m-2} - \text{guarded}_m$ and if $M_i^{(L(N_j^{A_{m-1} \cup B''}), L(N_l^{A_{m-1} \cup B''}))}(0^m)$ rejects then let \hat{w} be any string in $01\Sigma^{m-2} - \text{guarded}_m$.

By the same reason (i.e., item (iv) from earlier in the proof) that we can find a string z in the “otherwise” part of Case 2a we can find such a string \hat{w} . For each $m \leq p \leq \gamma$, set $A_p = A_{m-1} \cup B'' \cup \{\hat{w}\}$ and note that we again have achieved a type 2 cancellation for (h, e) . Add (h, e) to CAN . Note that we have maintained \otimes at the lengths $m, m+1, \dots, \gamma$. Go to stage $\gamma+1$.

This ends the construction of A

Note that for each $h_e = \langle i, j, k, l \rangle$ it holds that for all sufficiently large m the “ \geq ” in (iv) at the beginning of stage m fails (as all polynomials in m are $o(2^m)$). So the above construction ensures that any given pair (h, e) is eventually canceled, as once $h + e$ becomes the smallest among all uncanceled pairs it is clear it will eventually be canceled. Also, for every h we achieve either Goal 1 at some stage of the construction or we maintain \circledast in the construction of A at all lengths m for which $m = (p_h)^k$ for some k (where p_h is the h th prime).

The construction yields an oracle A such that any polynomial-time machine \widehat{M}_h either accepts at least one string $h_e = \langle i, j, k, l \rangle$ such that $L(N_j^A) \neq L(N_k^A)$ (in this case \widehat{M}_h does not accept a set of index quadruples having property 1 of Lemma 4.5) or else none of the machines (with corresponding oracles) whose quadruples are members of the set $L(\widehat{M}_h)$ accept the language D_h . (Note that in this latter case, D_h is in $P^{(NP^A \cap \text{coNP}^A, NP^A)}$ since for every h we maintain \circledast in the construction of A unless we have a type 1 cancellation of M_h .)

This proves that no polynomial-time machine can accept a set of index quadruples I having properties 1, 2, and 3 of Lemma 4.5. Thus, by Lemma 4.5, $(R_{1-tt}^{\mathcal{N}}(\text{NP}))^A$ has no $\leq_{1-tt}^{p,A}$ -complete sets. Finally, by our construction, A is clearly recursive (and, as is often the case regarding recursive oracles in complexity-theoretic constructions, one could make stronger claims regarding its time complexity). \blacksquare

For each A , since $(R_{1-tt}^{\mathcal{N}}(\text{NP}))^A \subseteq (R_{2-tt}^p(\text{NP}))^A$ and $\text{NP}^A \subseteq (R_{1-tt}^{\mathcal{N}}(\text{NP}))^A$, it follows that $(R_{1-tt}^{\mathcal{N}}(\text{NP}))^A$ clearly has $\leq_{2-tt}^{p,A}$ -complete sets. In particular, all sets $\leq_m^{p,A}$ -complete (or even $\leq_{1-tt}^{p,A}$ -complete) for NP^A are $\leq_{2-tt}^{p,A}$ -complete for $(R_{1-tt}^{\mathcal{N}}(\text{NP}))^A$. (We note in passing that it is also immediately clear that, for each A , $(R_{1-tt}^{\mathcal{N}}(\text{NP}))^A$ has $\leq_{1-tt}^{\mathcal{N},A}$ -complete sets, e.g., all sets $\leq_m^{p,A}$ -complete for NP^A .) We may now state the following from Theorem 4.6.

Corollary 4.7 *There is a recursive oracle A so that $(R_{1-tt}^{\mathcal{N}}(\text{NP}))^A$ has $\leq_{2-tt}^{p,A}$ -complete sets but has no $\leq_{1-tt}^{p,A}$ -complete sets.*

Weakening this, we have the following.

Corollary 4.8 *There is a recursive oracle A so that $(R_{1-tt}^{\mathcal{N}}(\text{NP}))^A$ has $\leq_T^{p,A}$ -complete sets but has no $\leq_m^{p,A}$ -complete sets.*

In summary, we showed that $R_{1-tt}^{\mathcal{N}}(\text{NP})$ distinguishes robust m-completeness from robust T-completeness. Indeed it distinguishes robust 1-tt-completeness from robust 2-tt-completeness. We conjecture that $\text{DP} \cap \text{coDP}$ will also distinguish robust 1-tt-completeness from 2-tt-completeness. However, note that this does not generalize to a claim that $\text{BH}_k \cap \text{coBH}_k$ (where BH_k is the k th level of the boolean hierarchy—see [CGH⁺88] for the definition of BH_k) distinguishes robust $k-1$ -tt-completeness from robust k -tt-completeness. In fact, it is clear that, for each $k \geq 2$, $\text{BH}_k \cap \text{coBH}_k$ robustly has 2-tt-complete sets (in fact,

for $k \geq 2$, it follows from the structure of the boolean hierarchy that all $\leq_m^{p,A}$ -complete sets for BH_{k-1} are $\leq_{2-tt}^{p,A}$ -complete for $\text{BH}_k \cap \text{coBH}_k$).

Acknowledgments

We are deeply indebted to Gerd Wechsung for both his kind encouragement and his valuable guidance. We thank Maren Hinrichs for enjoyable conversations, Alan Selman for helping us with the history of strong nondeterministic reductions, and an anonymous referee for helpful comments.

References

- [AH87] L. Adleman and M. Huang. Recognizing primes in random polynomial time. In *Proceedings of the 19th ACM Symposium on Theory of Computing*, pages 462–469. ACM Press, May 1987.
- [Amb86] K. Ambos-Spies. A note on complete problems for complexity classes. *Information Processing Letters*, 23:227–230, 1986.
- [BC93] D. Bovet and P. Crescenzi. *Introduction to the Theory of Complexity*. Prentice Hall, 1993.
- [BC97] R. Beigel and R. Chang. Commutative queries. In *Proceedings of the 5th Israeli Symposium on Theory of Computing and Systems*. IEEE Computer Society Press, June 1997. To appear.
- [BCS92] D. Bovet, P. Crescenzi, and R. Silvestri. A uniform approach to define complexity classes. *Theoretical Computer Science*, 104(2):263–283, 1992.
- [BDG95] J. Balcázar, J. Díaz, and J. Gabarró. *Structural Complexity I*. EATCS Monographs in Theoretical Computer Science. Springer-Verlag, 2nd edition, 1995.
- [BF96] H. Buhrman and L. Fortnow. Two queries. Technical Report 96-20, University of Chicago, Department of Computer Science, Chicago, IL, September 1996.
- [BGS75] T. Baker, J. Gill, and R. Solovay. Relativizations of the $P=?NP$ question. *SIAM Journal on Computing*, 4(4):431–442, 1975.
- [Bor94] B. Borchert. *Predicate Classes, Promise Classes, and the Acceptance Power of Regular Languages*. PhD thesis, Universität Heidelberg, Mathematisches Institut, Heidelberg, Germany, 1994.
- [CGH⁺88] J. Cai, T. Gundermann, J. Hartmanis, L. Hemachandra, V. Sewelson, K. Wagner, and G. Wechsung. The boolean hierarchy I: Structural properties. *SIAM Journal on Computing*, 17(6):1232–1252, 1988.

- [CHV93] J. Cai, L. Hemachandra, and J. Vyskoč. Promises and fault-tolerant database access. In K. Ambos-Spies, S. Homer, and U. Schöning, editors, *Complexity Theory*, pages 101–146. Cambridge University Press, 1993.
- [FK92] M. Fellows and N. Koblitz. Self-witnessing polynomial-time complexity and prime factorization. In *Proceedings of the 7th Structure in Complexity Theory Conference*, pages 107–110. IEEE Computer Society Press, June 1992.
- [GS88] J. Grollmann and A. Selman. Complexity measures for public-key cryptosystems. *SIAM Journal on Computing*, 17(2):309–335, 1988.
- [Gur83] Y. Gurevich. Algebras of feasible functions. In *Proceedings of the 24th IEEE Symposium on Foundations of Computer Science*, pages 210–214. IEEE Computer Society Press, November 1983.
- [HH88] J. Hartmanis and L. Hemachandra. Complexity classes without machines: On complete languages for UP. *Theoretical Computer Science*, 58:129–142, 1988.
- [HHH] E. Hemaspaandra, L. Hemaspaandra, and H. Hempel. A downward collapse within the polynomial hierarchy. *SIAM Journal on Computing*. To appear.
- [HHH97a] E. Hemaspaandra, L. Hemaspaandra, and H. Hempel. Query order in the polynomial hierarchy. In *Proceedings of the 11th Conference on Fundamentals of Computation Theory*. Springer Verlag *Lecture Notes in Computer Science #1279*, September 1997. To appear.
- [HHH97b] E. Hemaspaandra, L. Hemaspaandra, and H. Hempel. $R_{1-tt}^{SN}(NP)$ distinguishes robust many-one and Turing completeness. In *Proceedings of the 3rd Italian Conference on Algorithms and Complexity*, pages 49–60. Springer-Verlag *Lecture Notes in Computer Science #1203*, March 1997.
- [HHH97c] E. Hemaspaandra, L. Hemaspaandra, and H. Hempel. Translating equality downwards. Technical Report TR-657, University of Rochester, Department of Computer Science, Rochester, NY, April 1997.
- [HHW] L. Hemaspaandra, H. Hempel, and G. Wechsung. Query order. *SIAM Journal on Computing*. To appear.
- [HJV93] L. Hemaspaandra, S. Jain, and N. Vereshchagin. Banishing robust Turing completeness. *International Journal of Foundations of Computer Science*, 4(3):245–265, 1993.
- [HL94] S. Homer and L. Longpré. On reductions of NP sets to sparse sets. *Journal of Computer and System Sciences*, 48(2):324–336, 1994.
- [Kad88] J. Kadin. The polynomial time hierarchy collapses if the boolean hierarchy collapses. *SIAM Journal on Computing*, 17(6):1263–1282, 1988. Erratum appears in the same journal, 20(2):404.

- [KSW87] J. Köbler, U. Schöning, and K. Wagner. The difference and truth-table hierarchies for NP. *RAIRO Theoretical Informatics and Applications*, 21:419–435, 1987.
- [LLS75] R. Ladner, N. Lynch, and A. Selman. A comparison of polynomial time reducibilities. *Theoretical Computer Science*, 1(2):103–124, 1975.
- [LM96] J. Lutz and E. Mayordomo. Cook versus Karp-Levin: Separating completeness notions if NP is not small. *Theoretical Computer Science*, 164(1–2):123–140, 1996.
- [Lon82] T. Long. Strong nondeterministic polynomial-time reducibilities. *Theoretical Computer Science*, 21:1–25, 1982.
- [LY90] L. Longpré and P. Young. Cook Reducibility Is Faster than Karp reducibility in NP. *Journal of Computer and System Sciences*, 41(3):389–401, 1990.
- [OW91] M. Ogiwara and O. Watanabe. On polynomial-time bounded truth-table reducibility of NP sets to sparse sets. *SIAM Journal on Computing*, 20(3):471–483, June 1991.
- [Pap94] C. Papadimitriou. *Computational Complexity*. Addison-Wesley, 1994.
- [PY84] C. Papadimitriou and M. Yannakakis. The complexity of facets (and some facets of complexity). *Journal of Computer and System Sciences*, 28(2):244–259, 1984.
- [Reg89] K. Regan. Provable complexity properties and constructive reasoning. Manuscript, April 1989.
- [Ric89] C. Rich. Positive relativizations of the $P=?NP$ problem. *Journal of Computer and System Sciences*, 38(3):511–523, 1989.
- [Rog67] H. Rogers, Jr. *The Theory of Recursive Functions and Effective Computability*. McGraw-Hill, 1967.
- [Sel78] A. Selman. Polynomial time enumeration reducibility. *SIAM Journal on Computing*, 7(4):440–457, 1978.
- [Sel94a] V. Selivanov. Two refinements of the polynomial hierarchy. In *Proceedings of the 11th Annual Symposium on Theoretical Aspects of Computer Science*, pages 439–448. Springer-Verlag *Lecture Notes in Computer Science* #775, February 1994.
- [Sel94b] A. Selman. A taxonomy of complexity classes of functions. *Journal of Computer and System Sciences*, 48(2):357–381, 1994.
- [Sip82] M. Sipser. On relativization and the existence of complete sets. In *Proceedings of the 9th International Colloquium on Automata, Languages, and Programming*, pages 523–531. Springer-Verlag *Lecture Notes in Computer Science* #140, 1982.
- [SXB83] A. Selman, M. Xu, and R. Book. Positive relativizations of complexity classes. *SIAM Journal on Computing*, 12(3):565–579, 1983.
- [Wag90] K. Wagner. Bounded query classes. *SIAM Journal on Computing*, 19(5):833–846, 1990.

- [Wag97] K. Wagner. A note on parallel queries and the difference hierarchy. Technical Report 173, Universität Würzburg, Institut für Informatik, Würzburg, Germany, June 1997.
- [WT92] O. Watanabe and S. Tang. On polynomial-time Turing and many-one completeness in PSPACE. *Theoretical Computer Science*, 97(2):199–215, 1992.